Hamiltonian superoperators

1995 J. Phys. A: Math. Gen. 28 1681

(http://iopscience.iop.org/0305-4470/28/6/021)

J. Phys. A: Math. Gen. 28 (1995) 1681–1698. Printed in the UK

Hamiltonian superoperators

Xiaoping Xu

Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong†

Received 3 October 1994

Abstract. We present a theory of Hamiltonian superoperators associated with a Lie superalgebra and its modules. By using the free fermionic fields from physics, we establish a super version of the formal variational calculus introduced by Gel'fand and Dikii. Moreover, we prove that a super skew-symmetric matrix differential operator in our super formal variational calculus is a Hamiltonian operator if and only if its Schouten–Nijenhuis super-bracket is zero, when the characteristic of the base field is not two. Some interesting examples of Hamiltonian superoperators are also given.

1. Introduction

The 'formalization method' has been proved to be very powerful in many mathematical fields. Formal variational calculus was introduced by Gel'fand and Dikii [GDi1–2] in studying Hamiltonian systems related to certain nonlinear partial differential equations, such as the KdV equations. Invoking the variational derivatives, they found certain interesting Poisson structures. Moreover, Gel'fand and Dorfman [GDo] found more connections between Hamiltonian operators and certain algebraic structures. Balinskii and Novikov [BN] studied similar Poisson structures from another point of view. One of the algebraic structures appeared in [GDo] and [BN], which was called a 'Novikov algebra' by Osborn, was proved in [O1–3] to be closely related to rank-one Witt Lie algebras under certain conditions. One of the other structures in [GDo] was proved [X2] by this author to be equivalent to an associative algebra with a derivation under the unitary condition.

We observe that the formal variational calculus introduced by Gel'fand and Dikii [GDi1–2] can be rewritten in terms of free bosonic fields in physics. From an algebraic point of view, there should exist a formal variational calculus associated with free fermionic fields in physics. Our main purpose in this paper is to introduce a theory of Hamiltonian superoperators analogous to that given in [GDi1–2, GDo]. In fact, the calculus of Grassmannian variables exists in quantum many-particle systems (cf [NO], for example). Super-manifolds have been studied both by mathematicians and physicists for many years. In this sense, our study on Hamiltonian superoperators is natural. We also believe that the results in this paper could be useful for 'super-integrable systems' and 'super-symplectic geometry.'

Throughout this paper, we denote by $\mathbb{F}$ a field and denote by $\mathbb{Z}$ the ring of integers. All the vector spaces are over $\mathbb{F}$.

† Research supported by the Direct Alocation Grant 4083 DAG93/94 from HKUST.

0305-4470/95/061681+18$19.50

Let L be a Lie algebra and let M be an L-module. For $0 \leqslant q \in \mathbb{Z}$, a q-form $\omega : L^q = L \times \cdots \times L \to M$ is a skew-symmetric multilinear map. We denote the set of q-forms by $c^q(L, M)$. We take $c^0(L, M) = M$. The differential $d : c^q(L, M) \to c^{q+1}(L, M)$ is defined by

$$d\omega(a_1, \ldots, a_{q+1}) = \sum_{i=1}^{q+1}(-1)^{i+1} a_i \omega(a_1, \ldots, \check{a}_i, \ldots, a_{q+1})$$

$$+\sum_{i<j}(-1)^{i+j}\omega([a_i, a_j], a_1, \ldots, \check{a}_i, \ldots, \check{a}_j, \ldots, a_{q+1}) \tag{1.1}$$

for $a_l \in L$, where '$\check{a}_i$' means that a_i is omitted. A q-form ω is called *closed* if $d\omega = 0$.

Let Ω be a subspace of $c^1(L, M)$ satisfying $dM \subset \Omega$. Let $H : \Omega \to L$ be a linear operator (map) satisfying the skew-symmetry

$$\xi_1(H\xi_2) = -\xi_2(H\xi_1) \qquad \text{for } \xi_1, \xi_2 \in \Omega. \tag{1.2}$$

Moreover, we define $\omega_H \in c^2(H(\Omega), M)$ by

$$\omega_H(a_1, a_2) = \xi_2(a_1) \qquad \text{for } a_1, a_2 = H\xi_2 \in H(\Omega). \tag{1.3}$$

Note that (1.3) is well defined because of (1.2). The operator H is called *Hamiltonian* if $H(\Omega)$ forms a subalgebra of L and $d\omega_H = 0$.

In the formal variational calculus introduced by Gel'fand and Dikii [GDi1–2], one starts with the algebra A of polynomials of symbols $\{u_l^{(i)} \mid 0 \leqslant i \in \mathbb{Z}, l \in I\}$, where I is an index set. 'Differentiation with respect to x' is defined by the operator

$$\frac{d}{dx} = \sum_{l \in I}\sum_{i=0}^{\infty} u_l^{(i+1)} \frac{\partial}{\partial u_l^{(i)}}. \tag{1.4}$$

The partial variational derivatives $\delta/\delta u_l : A \to A$ are defined by

$$\frac{\delta}{\delta u_l} = \sum_{i=0}^{\infty}\left(-\frac{d}{dx}\right)^i \frac{\partial}{\partial u_l^{(i)}} \qquad l \in I. \tag{1.5}$$

Denote $\delta/\delta\bar{u} = \{\delta/\delta u_l \mid l \in I\}$. Then

$$\frac{\delta}{\delta\bar{u}} \circ \frac{d}{dx} = 0. \tag{1.6}$$

Set

$$\tilde{A} = A/(d/dx(A)). \tag{1.7}$$

The elements of $\tilde{A}$ are called *integrals*. Note that

$$[(du/dx)v]^{\sim} = -[u(dv/dx)]^{\sim} \qquad \text{for } u, v \in A. \tag{1.8}$$

Now let

$$L = \{\partial \in \operatorname{Der} A \mid [\partial, d/dx] = 0\}. \tag{1.9}$$

Then L is a Lie subalgebra of $\operatorname{Der} A$. We define the action of L on $\tilde{A}$ by

$$\partial\tilde{u} = (\partial u)^{\sim} \qquad \text{for } \partial \in L, u \in A. \tag{1.10}$$

The space $\tilde{A}$ becomes an L-module. Some Hamiltonian operators associated with (L, M) have been shown to be connected with certain very interesting nonlinear partial differential equations (cf [GDi1–2, GDo]). We observe that A can be viewed as a Fock space of the free bosonic fields

$$u_l(z) = \sum_{i=0}^{\infty} \frac{u_l^{(i)}}{(i+1)!} z^i + \sum_{i=0}^{\infty} (i+1)(i+1)! \frac{\partial}{\partial u_l^{(i)}} z^{-i-2} \qquad l \in I \tag{1.11}$$

which are viewed as 'operator-valued functions' (cf [FLM], for example). In this way, the operator d/dx coincides with the Virasoro operator $L(-1)$ (cf [FLM], for example). We shall study in this paper the Hamiltonian operators associated with the free fermionic fields. The paper is organized as follows.

In section 2, we introduce a general theory of Hamiltonian superoperators associated with a coloured Lie superalgebra and its modules. The super formal variational calculus is introduced in section 3. In section 4, we find the conditions of certain matrix differential operators to be Hamiltonian operators. Finally in section 5, we present some examples of Hamiltonian superoperators.

2. Closed 2-forms

In this section, we shall define 'closed q-forms' for a coloured Lie superalgebra and its modules. With a fixed closed 2-form, we connect a new Lie algebraic structure. Moreover, we set up the basic machinery for Hamiltonian superoperators.

Let Γ be an abelian group. Let $\vartheta(\cdot,\cdot) : \Gamma \times \Gamma \to \mathbb{F}^\times = \mathbb{F} \setminus \{0\}$ be a map satisfying

$$\vartheta(\alpha+\beta,\gamma) = \vartheta(\alpha,\gamma)\vartheta(\beta,\gamma) \qquad \vartheta(\alpha,\beta) = \vartheta(\beta,\alpha)^{-1} \qquad \text{for } \alpha,\ \beta,\ \gamma \in \Gamma. \tag{2.1}$$

A *coloured Lie superalgebra* $(L, \Gamma, \vartheta, [\cdot,\cdot])$ is a Γ-graded algebra $L = \bigoplus_{\alpha\in\Gamma} L_\alpha$ with the operation $[\cdot,\cdot]$ satisfying the super skew-symmetry

$$[x_1, x_2] = -\vartheta(\alpha,\beta)[x_2, x_1] \tag{2.2}$$

and the Jacobi identity

$$[x_1,[x_2,x_3]] + \vartheta(\alpha,\beta+\gamma)[x_2,[x_3,x_1]] + \vartheta(\alpha+\beta,\gamma)[x_3,[x_1,x_2]] = 0 \tag{2.3}$$

for $\alpha,\beta,\gamma \in \Gamma$; $x_1 \in L_\alpha, x_2 \in L_\beta, x_3 \in L_\gamma$. A *representation of* $(L, \Gamma, \vartheta, [\cdot,\cdot])$ is a map $\rho : L \to \mathrm{End}_{\mathbb{F}} M$ for some vector space M over $\mathbb{F}$ such that

$$\rho([x_1,x_2]) = \rho(x_1)\rho(x_2) - \vartheta(\alpha,\beta)\rho(x_2)\rho(x_1) \qquad \text{for } x_1 \in L_\alpha, x_2 \in L_\beta. \tag{2.4}$$

We simply denote

$$xu = \rho(x)u \qquad \text{for } x \in L,\ u \in M \tag{2.5}$$

and call M an *L-module*.

A *q-form of L with values in M* is a multi-linear map $\omega : L^q = L \times \cdots \times L \to M$ for which

$$\omega(x_1, x_2, \ldots, x_q) = -\vartheta(\alpha,\beta)\omega(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_q) \tag{2.6}$$

for $x_j \in L, x_i \in L_\alpha, x_{i+1} \in L_\beta$. We denote by $c^q(L,M)$ the set of q-forms. Moreover, we define a differential $d : c^q(L,M) \to c^{q+1}(L,M)$ by

$$\begin{aligned} d\omega(x_1, x_2, \ldots, x_{q+1}) &= \sum_{i=1}^{q+1} (-1)^{i+1} \vartheta(\alpha_1 + \cdots + \alpha_{i-1}, \alpha_i) x_i \omega(x_1, \ldots, \check{x}_i, \ldots, x_{q+1}) \\ &\quad + \sum_{i<j} (-1)^{i+j} \vartheta(\alpha_1 + \cdots + \alpha_{i-1}, \alpha_i)\vartheta(\alpha_1 + \cdots + \check{\alpha}_i + \cdots + \alpha_{j-1}, \alpha_j) \\ &\quad \times \omega([x_i, x_j], x_1, \ldots, \check{x}_i, \ldots, \check{x}_j, \ldots, x_{q+1}) \end{aligned} \tag{2.7}$$

for $\omega \in c^q(L,M), x_l \in L_{\alpha_l}, l = 1, \ldots, q+1$. A q-form ω is called *closed* if $d\omega = 0$.

Proposition 2.1. The differential d satisfies $d^2=0$.

Proof. For $\omega\in c^q(L,M)$, $x_1\in L_{\alpha_1},\ldots,x_{q+2}\in L_{\alpha_{q+2}}$, we have

$$
\begin{aligned}
&d^2\omega(x_1,\ldots,x_{q+2})\\
&=\sum_{i=1}^{q+2}(-1)^{i+1}\vartheta(\alpha_1+\cdots+\alpha_{i-1},\alpha_i)x_i d\omega(x_1,\ldots,\check{x}_i,\ldots,x_{q+2})\\
&\quad+\sum_{j<l}(-1)^{j+l}\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\\
&\quad\times\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)d\omega([x_j,x_l],x_1,\ldots,\check{x}_j,\ldots,\check{x}_l,\ldots,x_{q+2})\\
&=\sum_{j<i}(-1)^{i+j}\vartheta(\alpha_1+\cdots+\alpha_{i-1},\alpha_i)\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\\
&\quad\times x_j x_i\omega(x_1,\ldots,\check{x}_j,\ldots,\check{x}_i,\ldots,x_{q+2})\\
&\quad+\sum_{i<j}(-1)^{i+j+1}\vartheta(\alpha_1+\cdots+\alpha_{i-1},\alpha_i)\vartheta(\alpha_1+\cdots+\check{\alpha}_i+\cdots+\alpha_{j-1},\alpha_j)\\
&\quad\times x_j x_i\omega(x_1,\ldots,\check{x}_i,\ldots,\check{x}_j,\ldots,x_{q+2})\\
&\quad+\sum_{j<l<i}(-1)^{i+j+l+1}\vartheta(\alpha_1+\cdots+\alpha_{i-1},\alpha_i)\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\\
&\quad\times\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)\\
&\quad\times x_i\omega([x_j,x_l],x_1,\ldots,\check{x}_j,\ldots,\check{x}_l,\ldots,\check{x}_i,\ldots,x_{q+2})\\
&\quad+\sum_{j<i<l}(-1)^{i+j+l}\vartheta(\alpha_1+\cdots+\alpha_{i-1},\alpha_i)\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\\
&\quad\times\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\check{\alpha}_i+\cdots+\alpha_{l-1},\alpha_l)\\
&\quad\times x_i\omega([x_j,x_l],x_1,\ldots,\check{x}_j,\ldots,\check{x}_i,\ldots,\check{x}_l,\ldots,x_{q+2})\\
&\quad+\sum_{i<j<l}(-1)^{i+j+l-1}\vartheta(\alpha_1+\cdots+\alpha_{i-1},\alpha_i)\vartheta(\alpha_1+\cdots+\check{\alpha}_i+\cdots+\alpha_{j-1},\alpha_j)\\
&\quad\times\vartheta(\alpha_1+\cdots+\check{\alpha}_i+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)\\
&\quad\times x_i\omega([x_j,x_l],x_1,\ldots,\check{x}_i,\ldots,\check{x}_j,\ldots,\check{x}_l,\ldots,x_{q+2})\\
&\quad+\sum_{j<l}(-1)^{j+l}\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)\\
&\quad\times[x_j,x_l]\omega(x_1,\ldots,\check{x}_j,\ldots,\check{x}_l,\ldots,x_{q+2})\\
&\quad+\sum_{j<l<i}(-1)^{i+j+l}\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)\\
&\quad\times\vartheta(\alpha_1+\cdots+\alpha_{i-1},\alpha_i)x_i\omega([x_j,x_l],x_1,\ldots,\check{x}_j,\ldots,\check{x}_l,\ldots,\check{x}_i,\ldots,x_{q+2})
\end{aligned}
$$

$$+\sum_{j<i<l}(-1)^{i+j+l+1}\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)$$

$$\times\vartheta(\alpha_1+\cdots+\alpha_{i-1},\alpha_i)\vartheta(\alpha_l,\alpha_i)$$

$$\times x_i\omega([x_j,x_l],x_1,\ldots,\check{x}_j,\ldots,\check{x}_i,\ldots,\check{x}_l,\ldots,x_{q+2})$$

$$+\sum_{i<j<l}(-1)^{i+j+l}\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)$$

$$\times\vartheta(\alpha_1+\cdots+\alpha_{i-1},\alpha_i)\vartheta(\alpha_j+\alpha_l,\alpha_i)$$

$$\times x_i\omega([x_j,x_l],x_1,\ldots,\check{x}_i,\ldots,\check{x}_j,\ldots,\check{x}_l,\ldots,x_{q+2})$$

$$+\sum_{j<l<i}(-1)^{i+j+l}\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)$$

$$\times\vartheta(\alpha_1+\cdots+\check{\alpha}_j\cdots+\check{\alpha}_l\cdots+\alpha_{i-1},\alpha_i)$$

$$\times\omega([[x_j,x_l],x_i],x_1,\ldots,\check{x}_j,\ldots,\check{x}_l,\ldots,\check{x}_i,\ldots,x_{q+2})$$

$$+\sum_{j<i<l}(-1)^{i+j+l+1}\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)$$

$$\times\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{i-1},\alpha_i)$$

$$\times\omega([[x_j,x_l],x_i],x_1,\ldots,\check{x}_j,\ldots,\check{x}_i,\ldots,\check{x}_l,\ldots,x_{q+2})$$

$$+\sum_{i<j<l}(-1)^{i+j+l}\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)$$

$$\times\vartheta(\alpha_1+\cdots+\alpha_{i-1},\alpha_i)$$

$$\times\omega([[x_j,x_l],x_i],x_1,\ldots,\check{x}_i,\ldots,\check{x}_j,\ldots,\check{x}_l,\ldots,x_{q+2})$$

$$+\sum_{j<l<s<t}(-1)^{i+j+s+t}$$

$$\times\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)$$

$$\times\vartheta(\alpha_1+\cdots+\alpha_{s-1},\alpha_s)\vartheta(\alpha_1+\cdots+\check{\alpha}_s+\cdots+\alpha_{t-1},\alpha_t)$$

$$\times\omega([x_s,x_t],[x_j,x_l],x_1,\ldots,\check{x}_j,\ldots,\check{x}_l,\ldots,\check{x}_s,\ldots,\check{x}_t,\ldots,x_{q+2})$$

$$+\sum_{s<t<j<l}(-1)^{i+j+s+t}$$

$$\times\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)$$

$$\times\vartheta(\alpha_1+\cdots+\alpha_{s-1},\alpha_s)\vartheta(\alpha_1+\cdots+\check{\alpha}_s+\cdots+\alpha_{t-1},\alpha_t)$$

$$\times\vartheta(\alpha_j+\alpha_l,\alpha_s+\alpha_t)$$

$$\times\omega([x_s,x_t],[x_j,x_l],x_1,\ldots,\check{x}_s,\ldots,\check{x}_t,\ldots,\check{x}_j,\ldots.\check{x}_l,\ldots,x_{q+2})$$

$$+\sum_{j<s<l<t}(-1)^{i+j+s+t-1}$$

$$\times\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)$$
$$\times\vartheta(\alpha_1+\cdots+\alpha_{s-1},\alpha_s)\vartheta(\alpha_1+\cdots+\check{\alpha}_s+\cdots+\alpha_{t-1},\alpha_t)\vartheta(\alpha_l,\alpha_s)$$
$$\times\omega([x_s,x_t],[x_j,x_l],x_1,\ldots,\check{x}_j,\ldots,\check{x}_s,\ldots,\check{x}_l,\ldots,\check{x}_t,\ldots,x_{q+2})$$
$$+\sum_{s<j<t<l}(-1)^{i+j+s+t+1}$$
$$\times\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)$$
$$\times\vartheta(\alpha_1+\cdots+\alpha_{s-1},\alpha_s)\vartheta(\alpha_1+\cdots+\check{\alpha}_s+\cdots+\alpha_{t-1},\alpha_t)$$
$$\times\vartheta(\alpha_j+\alpha_l,\alpha_s)\vartheta(\alpha_l,\alpha_t)$$
$$\times\omega([x_s,x_t],[x_j,x_l],x_1,\ldots,\check{x}_s,\ldots,\check{x}_j,\ldots,\check{x}_t,\ldots,\check{x}_l,\ldots,x_{q+2})$$
$$+\sum_{j<s<t<l}(-1)^{i+j+s+t}$$
$$\times\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)$$
$$\times\vartheta(\alpha_1+\cdots+\alpha_{s-1},\alpha_s)\vartheta(\alpha_1+\cdots+\check{\alpha}_s+\cdots+\alpha_{t-1},\alpha_t)\vartheta(\alpha_l,\alpha_s+\alpha_t)$$
$$\times\omega([x_s,x_t],[x_j,x_l],x_1,\ldots,\check{x}_j,\ldots,\check{x}_s,\ldots,\check{x}_t,\ldots,\check{x}_l,\ldots,x_{q+2})$$
$$+\sum_{s<j<l<t}(-1)^{i+j+s+t}$$
$$\times\vartheta(\alpha_1+\cdots+\alpha_{j-1},\alpha_j)\vartheta(\alpha_1+\cdots+\check{\alpha}_j+\cdots+\alpha_{l-1},\alpha_l)$$
$$\times\vartheta(\alpha_1+\cdots+\alpha_{s-1},\alpha_s)\vartheta(\alpha_1+\cdots+\check{\alpha}_s+\cdots+\alpha_{t-1},\alpha_t)\vartheta(\alpha_j+\alpha_l,\alpha_s)$$
$$\times\omega([x_s,x_t],[x_j,x_l],x_1,\ldots,\check{x}_s,\ldots,\check{x}_j,\ldots,\check{x}_l,\ldots,\check{x}_t,\ldots,x_{q+2})\qquad(2.8)$$

which is equal to zero because of the following: the sum of the first two summations and the sixth is zero by (2.2); the sum of the third summation and the seventh is zero; the sum of the fourth summation and the eighth is zero; the sum of the fifth summation and the nineth is zero; the sum of the tenth to twelfth is zero by (2.3); the sum of the thirteenth summation and fourteenth, the sum of the fifteenth and sixteenth and the sum of the seventeenth and eighteenth are zero by (2.2). □

Let $\omega\in c^2(L,M)$. We define

$$\mathcal{H}_\alpha=\{(x,m)\in L_\alpha\times M\mid\omega(y,x)=ym\text{ for }y\in L\}\qquad\mathcal{H}=\sum_{\alpha\in\Gamma}\mathcal{H}_\alpha\subset L\times M.\qquad(2.9)$$

Now we suppose that ω is closed. Then for $(x_i,m_i)\in\mathcal{H}_{\alpha_i}$, $i=1,2,3$, we have

$$d\omega(x_1,x_2,x_3)=x_1\omega(x_2,x_3)-\vartheta(\alpha_1,\alpha_2)x_2\omega(x_1,x_3)+\vartheta(\alpha_1+\alpha_2,\alpha_3)x_3\omega(x_1,x_2)$$
$$-\omega([x_1,x_2],x_3)-\vartheta(\alpha_1,\alpha_2+\alpha_3)\omega([x_2,x_3],x_1)+\vartheta(\alpha_2,\alpha_3)\omega([x_1,x_3],x_2)$$
$$=-\vartheta(\alpha_2,\alpha_3)x_1x_3m_2+\vartheta(\alpha_1,\alpha_2+\alpha_3)x_2x_3m_1+\vartheta(\alpha_1+\alpha_2,\alpha_3)x_3\omega(x_1,x_2)$$
$$-\omega([x_1,x_2],x_3)-\vartheta(\alpha_1,\alpha_2+\alpha_3)[x_2,x_3]m_1+\vartheta(\alpha_2,\alpha_3)[x_1,x_3]m_2$$

$$= \vartheta(\alpha_1+\alpha_2,\alpha_3)x_3\omega(x_1,x_2) - \omega([x_1,x_2],x_3)$$
$$+\vartheta(\alpha_1,\alpha_2+ \ al_3)\vartheta(\alpha_2,\alpha_3)x_3x_2m_1 - \vartheta(\alpha_2,\alpha_3)\vartheta(\alpha_1,\alpha_3)x_3x_1m_2$$
$$= \vartheta(\alpha_1+\alpha_2,\alpha_3)x_3\omega(x_1,x_2) - \omega([x_1,x_2],x_3)$$
$$+\vartheta(\alpha_1+\alpha_2,\alpha_3)\vartheta(\alpha_1,\alpha_2)x_3\omega(x_2,x_1) - \vartheta(\alpha_1+\alpha_2,\alpha_3)x_3\omega(x_1,x_2)$$
$$= -\omega([x_1,x_2],x_3) - \vartheta(\alpha_1+\alpha_2,\alpha_3)x_3\omega(x_1,x_2). \tag{2.10}$$

Thus we have

$$\omega(x_3,[x_1,x_2]) = -\vartheta(\alpha_3,\alpha_1+\alpha_2)\omega([x_1,x_2],x_3) = x_3\omega(x_1,x_2) \tag{2.11}$$

by $d\omega = 0$. Hence $([x_1,x_2],\omega(x_1,x_2)) \in \mathcal{H}$. Therefore, we can define an operation $[\cdot,\cdot]$ on $\mathcal{H}$ by

$$[(x_1,m_1),(x_2,m_2)] = ([x_1,x_2],\omega(x_1,x_2)) = ([x_1,x_2],(x_1m_2 - \vartheta(\alpha_1,\alpha_2)x_2m_1)/2) \tag{2.12}$$

for $(x_i,m_i) \in \mathcal{H}_{\alpha_i}$.

Theorem 2.2. For a closed 2-form ω, the family $(\mathcal{H},\Gamma,\vartheta,[\cdot,\cdot])$ forms a coloured Lie superalgebra.

Proof. The super skew-symmetry follows by (2.12). For $(x_i,m_i) \in \mathcal{H}_{\alpha_i}$, $i = 1,2,3$, we have

$$[(x_1,m_1),[(x_2,m_2),(x_3,m_3)]]$$
$$= [(x_1,m_1),([x_2,x_3],\omega(x_2,x_3))]$$
$$= ([x_1,[x_2,x_3]],\omega(x_1,[x_2,x_3]))$$
$$= ([x_1,[x_2,x_3]],(\omega(x_1,[x_2,x_3]) + x_1\omega(x_2,x_3))/2)$$
$$= ([x_1,[x_2,x_3]],(-\vartheta(\alpha_1,\alpha_2+\alpha_3)\omega([x_2,x_3],x_1) + x_1\omega(x_2,x_3))/2). \tag{2.13}$$

Hence the left-hand side of the Jacobi identity (2.3) is equal to zero

$$[(x_1,m_1),[(x_2,m_2),(x_3,m_3)]] + \vartheta(\alpha_1,\alpha_2+\alpha_3)[(x_2,m_2),[(x_3,m_3),(x_1,m_1)]]$$
$$+\vartheta(\alpha_1+\alpha_2,\alpha_3)[(x_3,m_3),[(x_1,m_1),(x_2,m_2)]]$$
$$= \big([x_1,[x_2,x_3]] + \vartheta(\alpha_1,\alpha_2+\alpha_3)[x_2,[x_3,x_1]] + \vartheta(\alpha_1+\alpha_2,\alpha_3)[x_3,[x_1,x_2]],$$
$$(x_1\omega(x_2,x_3) + \vartheta(\alpha_1,\alpha_2+\alpha_3)x_2\omega(x_3,x_1) + \vartheta(\alpha_1+\alpha_2,\alpha_3)x_3\omega(x_1,x_2)$$
$$-\vartheta(\alpha_1,\alpha_2+\alpha_3)\omega([x_2,x_3],x_1)$$
$$-\vartheta(\alpha_1,\alpha_2+\alpha_3)\vartheta(\alpha_2,\alpha_1+\alpha_3)\omega([x_3,x_1],x_2)$$
$$-\vartheta(\alpha_1+\alpha_2,\alpha_3)\vartheta(\alpha_3,\alpha_1+\alpha_2)\omega([x_1,x_2],x_3))/2\big)$$
$$= (0,d\omega(x_1,x_2,x_3)/2)$$
$$= 0. \tag{2.14}$$

Therefore, the Jacobi identity is satisfied. □

Set

$$\mathcal{R}^l_\omega = \{x \in L \mid \omega(x, y) = 0 \text{ for any } y \in L\} \tag{2.15}$$

$$\mathcal{R}^r_\omega = \{x \in L \mid \omega(y, x) = 0 \text{ for any } y \in L\}. \tag{2.16}$$

In general, $\mathcal{R}^l_\omega \neq \mathcal{R}^r_\omega$. If $\mathcal{R}^l_\omega$ and $\mathcal{R}^r_\omega$ are Γ-graded, then $\mathcal{R}^l_\omega = \mathcal{R}^r_\omega$. We call ω *Γ-admissible* if $\mathcal{R}^l_\omega$ and $\mathcal{R}^r_\omega$ are Γ-graded. We assume now that ω is Γ-admissible and for any $u \in M$, $(L_\alpha, u) \bigcap \mathcal{H} \neq \emptyset$ for at most one $\alpha \in \Gamma$. Set

$$\mathcal{N}_\alpha = \{u \in M \mid (L_\alpha, u) \bigcap \mathcal{H} \neq \emptyset\} \qquad \mathcal{N} = \sum_{\alpha\in\Gamma} \mathcal{N}_\alpha. \tag{2.17}$$

Then $\mathcal{N}$ is Γ-graded. Furthermore, we define $\{\cdot, \cdot\}$ on $\mathcal{N}$ by

$$\{m_1, m_2\} = \omega(x_1, x_2) \qquad \text{for } (x_1, m_1), (x_2, m_2) \in \mathcal{H}. \tag{2.18}$$

This is a well defined operation since if $(x, m), (x, m') \in \mathcal{H}$, then $x - x' \in \mathcal{R}^l_\omega = \mathcal{R}^r_\omega$. Moreover, $(\mathcal{N}, \Gamma, \vartheta, \{\cdot, \cdot\})$ forms a coloured Lie superalgebra. We call ω a *super symplectic structure on* (L, M). The operation $\{\cdot, \cdot\}$ defined in (2.18) is called the *Poisson superbracket associated with this structure.*

Let Ω be a subspace of $c^1(L, M)$ such that $dM \subset \Omega$. Suppose that $H : \Omega \to L$ is a linear map. We call H *Γ-admissible* if

$$H(\Omega) = \bigoplus_{\alpha\in\Gamma} H(\Omega)_\alpha \qquad \text{where } H(\Omega)_\alpha = H(\Omega) \bigcap L_\alpha. \tag{2.19}$$

Moreover, H is called *super skew-symmetric* if

$$\xi_1(H\xi_2) = -\vartheta(\alpha_1, \alpha_2)\xi_2(H\xi_1) \qquad \text{where } H\xi_i \in (H(\Omega))_{\alpha_i}. \tag{2.20}$$

With a super skew-symmetric Γ-admissible linear map $H : \Omega \to L$, we connect a 2-form ω_H defined on $\operatorname{Im} H$ by

$$\omega_H(H\xi_1, H\xi_2) = \xi_2(H\xi_1) \qquad \text{for } \xi_1, \xi_2 \in \Omega. \tag{2.21}$$

Definition 2.3. We say that a super skew-symmetric Γ-admissible linear map $H : \Omega \to L$ is Hamiltonian if

(a) the subspace $\operatorname{Im} H$ of L is a subalgebra;
(b) the form ω_H is Γ-admissible and $d\omega_H \equiv 0$ on $H(\Omega)$.

Let H be a Hamiltonian operator. Moreover, we suppose that $H(dM)$ is Γ-graded. Then the space $\mathcal{H}$ defined in (2.9) becomes

$$\mathcal{H} = \{(Hdm, m) \mid m \in M\}. \tag{2.22}$$

Furthermore, we define an operation $\{\cdot, \cdot\}_H$ on M by

$$\{m_1, m_2\}_H = dm_2(Hdm_1) = (Hdm_1)(m_2) \qquad \text{for } m_1, m_2 \in M. \tag{2.23}$$

Then $(M, \Gamma, \vartheta, \{\cdot, \cdot\})$ forms a coloured Lie superalgebra, where

$$M = \bigoplus_{\alpha\in\Gamma} M_\alpha \qquad M_\alpha = \{m \in M \mid Hdm \in H(\Omega)_\alpha\}. \tag{2.24}$$

The map $Hd : M \to L$ is Lie superalgebra homomorphism from M to L.

Remark 2.4. Let $\mathcal{M}$ be a supermanifold defined in chapter 2 of [D]. Let $\mathcal{F}(\mathcal{M})$ be the set of scalar fields (analogues of differentiable functions of a manifold). Note that $\mathcal{F}(\mathcal{M})$ forms a super-commutative algebra (cf [D]). The set $\mathcal{X}(\mathcal{M})$ of contravariant vector fields of $\mathcal{M}$ forms a Lie superalgebra, and $\mathcal{F}(\mathcal{M})$ is module of $\mathcal{X}(\mathcal{M})$ (cf [D]). A Hamiltonian operator associated with $(\mathcal{X}(\mathcal{M}), \mathcal{F}(\mathcal{M}))$ provides a super-Lie–Poisson structure over $\mathcal{M}$. We would like to present a detailed study of this in our future work.

3. Super formal variational calculus

In this section, we shall present a super version of the formal variational calculus introduced by Gel'fand and Dikii [GDi1–2]. Our idea follows the observation that the formal variational calculus can be written in terms of the well known free bosonic fields in physics. We want to establish an analogous theory associated with the well known free fermionic fields in physics.

Let $\mathcal{S}$ be a vector space with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$. Set

$$\hat{\mathcal{S}} = \mathbb{F}[t]t^{\frac{1}{2}} \otimes_{\mathbb{F}} \mathcal{S} \tag{3.1}$$

where t is an indeterminant. Denote

$$h(n) = t^n \otimes h \qquad \text{for } h \in \mathcal{S}, l \in \mathbb{Z} + \tfrac{1}{2}. \tag{3.2}$$

We extend $\langle\cdot,\cdot\rangle$ to $\hat{\mathcal{S}}$ by

$$\langle h(m), h'(n)\rangle = \delta_{m+n,0}\langle h, h'\rangle \qquad \text{for } h, h' \in H;\ m, n \in \mathbb{Z} + \tfrac{1}{2}. \tag{3.3}$$

Let $A_{\mathcal{S}}$ be the free algebra generated by $\hat{\mathcal{S}}$ and let J be the ideal of $A_{\mathcal{S}}$ generated by

$$\{uv + vu - \langle u, v\rangle \mid u, v \in \hat{\mathcal{S}}\}. \tag{3.4}$$

Then we have a Clifford algebra

$$V = A_{\mathcal{S}}/J. \tag{3.5}$$

Set

$$\mathbb{N}' = \{l \in \mathbb{Z} + \tfrac{1}{2} \mid l > 0\} \qquad \hat{\mathcal{S}}_{\pm} = \{h(l) \mid 0 < \pm l \in \mathbb{N}'\}. \tag{3.6}$$

We denote by $V^{\pm}$ the subalgebra generated by $\hat{\mathcal{S}}_{\pm}$. Then $V^{\pm}$ are the exterior algebras generated by $\hat{\mathcal{S}}_{\pm}$. Moreover

$$V = V^- V^+. \tag{3.7}$$

Let $\mathbb{F}v_0$ be a one-dimensional V^+-module such that

$$x v_0 = 0 \qquad \text{for } x \in \hat{\mathcal{S}}^+. \tag{3.8}$$

We form an induced module

$$\mathcal{U} = V \otimes_{V^+} \mathbb{F}v_0. \tag{3.9}$$

By (3.7),

$$\mathcal{U} \cong V^- \qquad \text{as vector spaces.} \tag{3.10}$$

We identify $y \otimes v_0$ with y for $y \in V^-$. In this way, we can view the elements of $\hat{\mathcal{S}}_+$ as 'super-derivations' of the exterior algebra V^-. Set

$$\mathcal{U}_0 = \operatorname{span}\{h_1(-l_1)\cdots h_{2k}(-l_{2k}) \mid 0 \leqslant k \in \mathbb{Z}, 0 < l_i \in \mathbb{N}'\} \tag{3.11}$$

$$\mathcal{U}_1 = \operatorname{span}\{h_1(-l_1)\cdots h_{2k+1}(-l_{2k+1}) \mid 0 \leqslant k \in \mathbb{Z}, 0 < l_i \in \mathbb{N}'\}. \tag{3.12}$$

Then $\mathcal{U}(\equiv V^-)$ is a $\mathbb{Z}_2$-graded algebra, where $\mathbb{Z}_2 = \mathbb{Z}/(2)$ and we use the notion $\mathbb{Z}_2 = \{0, 1\}$ when the context is clear. In fact

$$xy = (-1)^{ij} yx \qquad \text{for } x \in \mathcal{U}_i,\ y \in \mathcal{U}_j. \tag{3.13}$$

Let $\{s_i \mid i \in I\}$ be an orthnormal basis of $\mathcal{S}$, where I is an index set. For $i \in \mathbb{Z}_2$, we set

$$L_{i+1} = \left\{ \sum_{j\in I} \sum_{l\in \mathbb{N}'} u_{j,l} s_j(l) \mid u_{j,l} \in \mathcal{U}_i \right\}. \tag{3.14}$$

Note that for $\partial \in L_i$, $u \in \mathcal{U}_j$, $v \in \mathcal{U}$,

$$\partial(uv) = \partial(u)v + (-1)^{ij} u\partial(v). \tag{3.15}$$

Thus $L = L_0 \oplus L_1$ is a set of super-derivations on $\mathcal{U}$. For $\partial_1 = \sum_{j\in I}\sum_{l\in\mathbb{N}'} u^1_{j,l} s_j(l) \in L_{i_1}$, $\partial_2 = \sum_{j\in I}\sum_{l\in\mathbb{N}'} u^2_{j,l} s_j(l) \in L_{i_2}$, we define

$$[\partial_1, \partial_2] = \sum_{j,p\in I} \sum_{l,q\in\mathbb{N}'} (u^1_{p,q} s_p(q)(u^2_{j,l}) - (-1)^{i_1 i_2} u^2_{p,q} s_p(q)(u^1_{j,l})) s_j(l) \in L_{i_1+i_2}. \tag{3.16}$$

It can be proved that acting on $\mathcal{U}$

$$[\partial_1, \partial_2] = \partial_1\partial_2 - (-1)^{i_1 i_2}\partial_2\partial_1. \tag{3.17}$$

Define $\vartheta : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{F}^\times$ by $\vartheta(i, j) = (-1)^{ij}$. Then $(L, \mathbb{Z}_2, \vartheta, [\cdot,\cdot])$ forms a Lie superalgebra.

Set

$$D = \sum_{j\in I}\sum_{l\in\mathbb{N}'} \left(l + \tfrac{1}{2}\right) s_j(-l-1) s_j(l) \in L_0. \tag{3.18}$$

A direct verification shows that

$$D\left(h\left(-l - \tfrac{1}{2}\right)\right) = (l+1)h\left(-l - \tfrac{3}{2}\right) \qquad \left[D, h\left(l + \tfrac{1}{2}\right)\right] = -lh\left(l - \tfrac{1}{2}\right) \tag{3.19}$$

for $h \in \mathcal{S}, l \in \mathbb{N}'$ (cf [FFR, T, X2]).

Remark 3.1.

(a) The operator D is an analogue of d/dx in the formal varational calculus [GDi1–2].

(b) We can view $\mathcal{U}$ as a Fock space of the following free fermionic fields:

$$s_j(z) = \sum_{m\in\mathbb{Z}} s_j\left(m + \tfrac{1}{2}\right) z^{-m-1} \qquad j \in I \tag{3.20}$$

which can be viewed as 'operator valued functions' on $\mathcal{U}$ (cf [FFR, T, X1], for example). In this way, our operator D coincides with the Virasoro operator $L(-1)$ (cf [FFR, T, X1], for example).

Lemma 3.2. For $\partial = \sum_{j\in I}\sum_{l\in\mathbb{N}'} u_{j,l} s_j(l) \in L$, $[\partial, D] = 0$ if and only if

$$u_{j,n+\frac{1}{2}} = \frac{D^n}{n!}(u_{j,\frac{1}{2}}) \qquad 0 \leqslant n \in \mathbb{Z}. \tag{3.21}$$

Proof. Note that

$$[\partial, D] = \sum_{j\in I}\sum_{0\leqslant n\in\mathbb{Z}} (D(u_{j,n+\frac{1}{2}}) - (n+1)u_{j,n+1+\frac{1}{2}}) s_j(n + \tfrac{1}{2}). \tag{3.22}$$

Thus $[\partial, D] = 0$ is equivalent to

$$D(u_{j,n+\frac{1}{2}}) = (n+1)u_{j,n+1+\frac{1}{2}}. \tag{3.23}$$

Then (3.21) follows by induction on n. □

Set

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \subset \mathcal{U}^I \qquad \mathcal{L}_i = (\mathcal{U}_i)^I. \tag{3.24}$$

For any $\bar{u} = \{u_i \mid i \in I\} \in \mathcal{L}$, we let

$$\partial_{\bar{u}} = \sum_{j\in I} \sum_{0 \leqslant n \in \mathbb{Z}} \frac{D^n}{n!}(u_j) s_j(n + \tfrac{1}{2}) \in L. \tag{3.25}$$

Then $[\partial_{\bar{u}}, D] = 0$.

For $\bar{u} = \{u_i\} \in \mathcal{L}_i$, $\bar{v} = \{v_i\} \in \mathcal{L}_j$,

$$\begin{aligned}
[\partial_{\bar{u}}, \partial_{\bar{v}}] = \sum_{p,q\in I} \sum_{0 \leqslant m,n \in \mathbb{Z}} \Bigg[& \frac{D^m(u_p)}{m!} s_p\left(m + \tfrac{1}{2}\right)\left(\frac{D^n(v_q)}{n!}\right) \\
& -(-1)^{(i+1)(j+1)} \frac{D^m(v_p)}{m!} s_p\left(m + \tfrac{1}{2}\right)\left(\frac{D^n(u_q)}{n!}\right)\Bigg] s_q\left(n + \tfrac{1}{2}\right) \\
= \sum_{p,q\in I} \sum_{0 \leqslant m,n \in \mathbb{Z}} & \frac{D^n}{n!}\Bigg(\frac{D^m(u_p)}{m!} s_p\left(m + \tfrac{1}{2}\right)(v_q) \\
& -(-1)^{(i+1)(j+1)} \frac{D^m(v_p)}{m!} s_p\left(m + \tfrac{1}{2}\right)(u_q)\Bigg) s_q\left(n + \tfrac{1}{2}\right) \\
= \partial_{\bar{w}} &
\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}
\bar{w} = \Bigg\{ \sum_{p\in I} \sum_{0 \leqslant m \in \mathbb{Z}} \Bigg(& \frac{D^m(u_p)}{m!} s_p\left(m + \tfrac{1}{2}\right)(v_q) \\
& -(-1)^{(i+1)(j+1)} \frac{D^m(v_p)}{m!} s_p\left(m + \tfrac{1}{2}\right)(u_q)\Bigg) \Bigg| \, q \in I \Bigg\}.
\end{aligned} \tag{3.27}$$

Thus if we define

$$[\bar{u}, \bar{v}] = \bar{w} \tag{3.28}$$

then $(\mathcal{L}, \mathbb{Z}_2, \vartheta, [\cdot, \cdot])$ forms a Lie superalgebra.

Next we define variational operators on $\mathcal{U}$:

$$\delta_i = \sum_{m=0}^{\infty} \frac{(-D)^m}{m!} \circ s_i\left(m + \tfrac{1}{2}\right) \qquad \bar{\delta} = \{\delta_i \mid i \in I\}. \tag{3.29}$$

Lemma 3.4. For any $u \in \mathcal{U}\hat{S}_-$,

$$\bar{\delta}(u) = 0 \Longleftrightarrow u = D(v) \qquad \text{for some } v \in \mathcal{U}. \tag{3.30}$$

Proof. Set

$$\bar{s} = \{s_i(-\tfrac{1}{2})\} \qquad D_0 = \partial_{\bar{s}} = \sum_{i\in I}\sum_{m=0}^{\infty} s_i\left(-m-\tfrac{1}{2}\right) s_i\left(m+\tfrac{1}{2}\right) \in \mathcal{L}_0. \tag{3.31}$$

For any $h_1(-l_1)\cdots h_p(-l_p) \in \mathcal{U}$, we have

$$D_0(h_1(-l_1)\cdots h_p(-l_p)) = ph_1(-l_1)\cdots h_p(-l_p). \tag{3.32}$$

Thus $D_0(\mathcal{U}\hat{S}_-) = \mathcal{U}\hat{S}_-$ and $D_0|_{\mathcal{U}\hat{S}_-}$ is a linear isomorphism. If $\vec{\delta}(u) = 0$, then

$$\delta_i(u) = \sum_{m=0}^{\infty} \frac{(-D)^m}{m!}\left(s_i\left(m+\tfrac{1}{2}\right)(u)\right) = 0 \qquad \text{for } i \in I. \tag{3.33}$$

Therefore

$$\begin{aligned}
s_i\left(-\tfrac{1}{2}\right) s_i\left(\tfrac{1}{2}\right)(u) &= \sum_{m=1}^{\infty} s_i\left(-\tfrac{1}{2}\right) D\frac{(-D)^{m-1}}{m!}\left(s_i\left(m+\tfrac{1}{2}\right)(u)\right)\\
&= D\left(\sum_{m=1}^{\infty} s_i\left(-\tfrac{1}{2}\right)\frac{(-D)^{m-1}}{m!}\left(s_i\left(m+\tfrac{1}{2}\right)(u)\right)\right) - s_i\left(-1-\tfrac{1}{2}\right)s_i\left(1+\tfrac{1}{2}\right)(u)\\
&\quad + \sum_{m=1}^{\infty} s_i\left(-1-\tfrac{1}{2}\right) D\frac{(-D)^{m-1}}{(m+1)!}\left(s_i\left(m+1+\tfrac{1}{2}\right)(u)\right)\\
&= \cdots\\
&= -\sum_{m=1}^{\infty} s_i\left(-m-\tfrac{1}{2}\right) s_i\left(m+\tfrac{1}{2}\right)(u) + D(w) \qquad \text{for some } w \in \mathcal{U}
\end{aligned} \tag{3.34}$$

where the last step follows by induction and the fact that $s_i\left(m+\frac{1}{2}\right)(u) = 0$ for sufficiently large m. Thus

$$D_0(u) = D(w). \tag{3.35}$$

By (3.19) and (3.32)

$$[D_0, D] = 0. \tag{3.36}$$

Hence $u = D((D_0|_{\mathcal{U}\hat{S}_-})^{-1}(w)) \in D(\mathcal{U})$. Conversely, if $u = D(v)$,

$$\begin{aligned}
\delta_i(D(v)) &= \sum_{m=0}^{\infty} \frac{(-D)^m}{m!}\left(s_i\left(m+\tfrac{1}{2}\right)D(v)\right)\\
&= -\sum_{m=0}^{\infty} \frac{(-D)^{m+1}}{m!}\left(s_i\left(m+\tfrac{1}{2}\right)(v)\right)\\
&\quad + \sum_{m=1}^{\infty} \frac{(-D)^m}{(m-1)!}\left(s_i\left(m-1+\tfrac{1}{2}\right)(v)\right)\\
&= 0.
\end{aligned} \tag{3.37}$$

□

Now we let

$$\tilde{\mathcal{U}} = \mathcal{U}/D(\mathcal{U}). \tag{3.38}$$

We define an action of $\mathcal{L}$ on $\tilde{\mathcal{U}}$ by

$$\begin{aligned}\bar{u}(\tilde{w}) &= \partial_{\bar{u}}(w) + D(\mathcal{U}) \\ &= \sum_{i\in I}\sum_{m=0}^{\infty}\frac{D^m(u_i)}{m!}s_i\left(m+\tfrac{1}{2}\right)(w) + D(\mathcal{U}) \\ &= \sum_{i\in I} u_i\delta_i(w) + D(\mathcal{U}) \\ &= \sum_{i\in I}(u_i\delta_i(w))^{\sim}.\end{aligned} \tag{3.39}$$

This is well defined since $[\partial_{\bar{u}}, D] = 0$. Thus $\tilde{\mathcal{U}}$ forms an $\mathcal{L}$-module. Furthermore, we set

$$\Omega = \{\bar{\xi} = \{\xi_i\} \in \mathcal{U}^I \mid \text{only finite number of } \xi_i \neq 0\}. \tag{3.40}$$

For any $\bar{\xi} \in \Omega$, $\bar{u} \in \mathcal{L}$, we define

$$\bar{\xi}(\bar{u}) = \sum_{i\in I}(u_i\xi_i)^{\sim}. \tag{3.41}$$

Then $\Omega \subset c^1(\mathcal{L}, \tilde{\mathcal{U}})$. Note that by (3.39)

$$d(\tilde{w}) = \bar{\delta}(w) \in \Omega \qquad \text{for } \tilde{w} \in \mathcal{U} \tag{3.42}$$

where (3.30) implies that the map $\bar{\delta} : \tilde{\mathcal{U}} \to \Omega$ is well defined. Hence $d(\tilde{\mathcal{U}}) \in \Omega$.

4. Hamiltonian operators in super formal variational calculus

In this section, we study Hamiltonian operators in our super formal variational calculus. In particular, we give a super version of Schouten–Nijenhuis bracket, whose nullity is the condition for certain matrix differential operators to be Hamiltonian.

Note that as sets, $\Omega \subset \mathcal{L}$. We let

$$\Omega_i = \Omega \bigcap \mathcal{L}_i \qquad \text{for } i \in \mathbb{Z}_2. \tag{4.1}$$

Suppose that $H : \Omega \to \mathcal{L}$ is a linear map as follows: for $\bar{\xi} \in \Omega_i$, $i \in \mathbb{Z}_2$

$$(H\bar{\xi})_p = \sum_{q\in I} H^i_{p,q}\xi_q \qquad \text{where } H^i_{p,q} = \sum_{l=0}^{n(i,p,q)} a^i_{p,q,l}D^l \quad \text{with } a^i_{p,q,l} \in \mathcal{U}_\iota,\ \iota \in \mathbb{Z}_2. \tag{4.2}$$

Such an H is called a *matrix differential operator of type* ι. Moreover, $H(\Omega)$ is a $\mathbb{Z}_2$-graded subspace. Furthermore, the super skew-symmetry is equivalent to

$$\sum_{l=0}^{n(0,p,q)}(-D)^l a^0_{p,q,l} = (-1)^\iota \sum_{l=0}^{n(0,q,p)} a^0_{q,p,l}D^l \qquad a^0_{p,q,l} = (-1)^{\iota+1}a^1_{p,q,l}. \tag{4.3}$$

Let $H : \Omega \to \mathcal{L}$ be a super skew-symmetric matrix differential operator. We want to find the condition for H to be a Hamiltonian operator. For $\bar{\xi} \in \Omega_i$, we define a linear map $(D_H\bar{\xi}) : \mathcal{L} \to \mathcal{L}$ by

$$(D_H\bar{\xi})(\bar{\eta}) = (D_H\bar{\xi})\bar{\eta} \qquad \text{where } (D_H\bar{\xi})_{p,q} = \sum_{t\in I}\sum_{0\leqslant l,m\in\mathbb{Z}} s_q\left(m+\tfrac{1}{2}\right)(a^i_{p,t,l})D^l(\xi_t)\frac{D^m}{m!} \tag{4.4}$$

for $\bar{\eta} \in \Omega$. Now for $\bar{\xi} \in \Omega_i, \bar{\eta} \in \Omega_j$, we have

$$\begin{aligned}
\partial_{H\bar{\xi}}(H\bar{\eta})_q &= \sum_{t\in I}\sum_{l=0}^{\infty} \partial_{H\bar{\xi}}(a^j_{q,t,l} D^l \eta_t) \\
&= \sum_{t\in I}\sum_{l=0}^{\infty} \partial_{H\bar{\xi}}(a^j_{q,t,l}) D^l \eta_t + \sum_{t\in I}\sum_{l=0}^{\infty}(-1)^{\iota(i+\iota)} a^j_{q,t,l} \partial_{H\bar{\xi}}(D^l \eta_t) \\
&= \sum_{p,t\in I}\sum_{l,m=0}^{\infty} \frac{D^m (H\bar{\xi})_p}{m!} s_p\left(m+\tfrac{1}{2}\right)(a^j_{q,t,l}) D^l \eta_t \\
&\quad + \sum_{t\in I}\sum_{l=0}^{\infty}(-1)^{\iota(i+\iota)} a^j_{q,t,l} D^l \partial_{H\bar{\xi}}(\eta_t) \\
&= \sum_{p,t\in I}\sum_{l,m=0}^{\infty} (-1)^{(\iota+i+1)(\iota+j)} s_p\left(m+\tfrac{1}{2}\right)(a^j_{q,t,l}) D^l \eta_t \frac{D^m (H\bar{\xi})_p}{m!} \\
&\quad + (-1)^{i+\iota} \sum_{t\in I}\sum_{l=0}^{\infty} a^{\iota+i+j}_{q,t,l} D^l \partial_{H\bar{\xi}}(\eta_t).
\end{aligned} \tag{4.5}$$

Thus

$$\partial_{H\bar{\xi}}(H\bar{\eta}) = (-1)^{(\iota+i+1)(\iota+j)}(D_H\bar{\eta})(H\bar{\xi}) + (-1)^{i+\iota} H\partial_{H\bar{\xi}}(\bar{\eta}). \tag{4.6}$$

Furthermore, for $\bar{\xi}_i \in \Omega_{j_i}, i = 1, 2, 3$,

$$\begin{aligned}
(H\bar{\xi}_1)\omega_H(H\bar{\xi}_2, H\bar{\xi}_3) &= \partial_{H\bar{\xi}_1}[\bar{\xi}_3(H\bar{\xi}_2)] \\
&= (-1)^{(\iota+j_1)(j_2+\iota+1)}(\partial_{H\bar{\xi}_1}\bar{\xi}_3)(H\bar{\xi}_2) + \bar{\xi}_3(\partial_{H\bar{\xi}_1} H\bar{\xi}_2) \\
&= -(-1)^{(\iota+j_3)(\iota+j_2)+\iota+j_1}\bar{\xi}_2(H\partial_{H\bar{\xi}_1}\bar{\xi}_3) + \omega_H(\partial_{H\bar{\xi}_1} H\bar{\xi}_2, H\bar{\xi}_3)
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&\omega_H([H\bar{\xi}_1, H\bar{\xi}_2], H\bar{\xi}_3) \\
&\quad = \omega_H(\partial_{H\bar{\xi}_1} H\bar{\xi}_2, H\bar{\xi}_3) - (-1)^{(\iota+j_1)(\iota+j_2)}\bar{\xi}_3(\partial_{H\bar{\xi}_2} H\bar{\xi}_1) \\
&\quad = \omega_H(\partial_{H\bar{\xi}_1} H\bar{\xi}_2, H\bar{\xi}_3) - (-1)^{j_1+\iota}\bar{\xi}_3((D_H\bar{\xi}_1)(H\bar{\xi}_2)) \\
&\quad\quad - (-1)^{(\iota+j_1+1)(\iota+j_2)}\bar{\xi}_3(H\partial_{H\bar{\xi}_2}\bar{\xi}_1)
\end{aligned} \tag{4.8}$$

where we have extended the definition of ω_H to $\mathcal{L} \times H(\Omega)$. Therefore, by (2.7) and (4.7), (4.8), the equation $d\omega_H(H\bar{\xi}_1, H\bar{\xi}_2, H\bar{\xi}_3) = 0$ is equivalent to

$$\begin{aligned}
&(-1)^{j_1}\bar{\xi}_3((D_H\bar{\xi}_1)H\bar{\xi}_2) + (-1)^{j_2+(j_1+\iota, j_2+j_3)}\bar{\xi}_1((D_H\bar{\xi}_2)H\bar{\xi}_3) \\
&\quad + (-1)^{j_3+(j_3+\iota, j_1+j_2)}\bar{\xi}_2((D_H\bar{\xi}_3)H\bar{\xi}_1) = 0.
\end{aligned} \tag{4.9}$$

Theorem 4.1. A matrix differential operator H of the form (4.2) is a Hamiltonian operator if and only if equations (4.3) and (4.9) hold.

Proof. By the above argument, we only need to prove that (4.3) and (4.9) imply that $H(\Omega)$ is a subalgebra of $\mathcal{L}$. Again we let $\bar{\xi}_i \in \Omega_i$, $i = 1, 2, 3$. Note that

$$
\begin{aligned}
(H\bar{\xi}_1)\omega_H(H\bar{\xi}_2, H\bar{\xi}_3) &= (H\bar{\xi}_1)[\bar{\xi}_3(H\bar{\xi}_2)] \\
&= (-1)^{(j_1+\iota)(j_2+\iota+1)}(\partial_{H\bar{\xi}_1}\bar{\xi}_3)(H\bar{\xi}_2) + \bar{\xi}_3(\partial_{H\bar{\xi}_1}H\bar{\xi}_2) \\
&\overset{(4.6)}{=} (-1)^{(j_1+\iota)(j_2+\iota+1)}(\partial_{H\bar{\xi}_1}\bar{\xi}_3)(H\bar{\xi}_2) + (-1)^{(j_1+\iota+1)(j_2+\iota)}\bar{\xi}_3[(D_H\bar{\xi}_2)H\bar{\xi}_1] \\
&\quad + (-1)^{j_1+\iota}\bar{\xi}_3(H\partial_{H\bar{\xi}_1}\bar{\xi}_2) \\
&= (-1)^{(j_1+\iota)(j_2+\iota+1)}(\partial_{H\bar{\xi}_1}\bar{\xi}_3)(H\bar{\xi}_2) + (-1)^{(j_1+\iota+1)(j_2+\iota)}\bar{\xi}_3[(D_H\bar{\xi}_2)H\bar{\xi}_1] \\
&\quad - (-1)^{j_1+\iota+(j_1+j_2)(j_3+\iota)}(\partial_{H\bar{\xi}_1}\bar{\xi}_2)(H\bar{\xi}_3).
\end{aligned}
\tag{4.10}
$$

On the other hand

$$
\begin{aligned}
(H\bar{\xi}_1)\omega_H(H\bar{\xi}_2, H\bar{\xi}_3) &= (-1)^{(j_2+\iota)(j_3+\iota)+1}(H\bar{\xi}_1)\omega_H(H\bar{\xi}_3, H\bar{\xi}_2) \\
&= (-1)^{(j_2+\iota)(j_3+\iota)+1+(j_1+\iota)(j_3+\iota+1)}(\partial_{H\bar{\xi}_1}\bar{\xi}_2)(H\bar{\xi}_3) + (-1)^{(j_2+\iota)(j_3+\iota)+1+(j_1+\iota+1)(j_3+\iota)} \\
&\quad \times\bar{\xi}_2[(D_H\bar{\xi}_3)H\bar{\xi}_1] + (-1)^{(j_2+\iota)(j_3+\iota)+j_1+\iota+(j_1+j_3)(j_2+\iota)}(\partial_{H\bar{\xi}_1}\bar{\xi}_3)(H\bar{\xi}_2) \\
&= -(-1)^{(j_1+j_2)(j_3+\iota)+j_1+\iota}(\partial_{H\bar{\xi}_1}\bar{\xi}_2)(H\bar{\xi}_3) + (-1)^{(j_1+j_2+1)(j_3+\iota)+1}\bar{\xi}_2[(D_H\bar{\xi}_3)H\bar{\xi}_1] \\
&\quad + (-1)^{(j_1+\iota)(j_2+\iota+1)}(\partial_{H\bar{\xi}_1}\bar{\xi}_3)(H\bar{\xi}_2).
\end{aligned}
\tag{4.11}
$$

Thus

$$
(-1)^{(j_1+\iota+1)(j_2+\iota)}\bar{\xi}_3[(D_H\bar{\xi}_2)H\bar{\xi}_1] = (-1)^{(j_1+j_2+1)(j_3+\iota)+1}\bar{\xi}_2[(D_H\bar{\xi}_3)H\bar{\xi}_1] \tag{4.12}
$$

or equivalently

$$
\bar{\xi}_3[(D_H\bar{\xi}_2)H\bar{\xi}_1] = (-1)^{(j_1+\iota)(j_2+\iota)+(j_1+j_2)(j_3+\iota)+j_2+j_3+1}\bar{\xi}_2[(D_H\bar{\xi}_3)H\bar{\xi}_1]. \tag{4.13}
$$

Furthermore

$$
\begin{aligned}
&\bar{\xi}_3[(-1)^{(j_1+\iota+1)(j_2+\iota)}(D_H\bar{\xi}_2)(H\bar{\xi}_1) - (-1)^{(j_1+\iota)(j_2+\iota)+(j_2+\iota+1)(j_1+\iota)}(D_H\bar{\xi}_1)(H\bar{\xi}_2)] \\
&\quad = (-1)^{(j_1+\iota+1)(j_2+\iota)}\bar{\xi}_3[(D_H\bar{\xi}_2)(H\bar{\xi}_1)] - (-1)^{j_1+\iota}\bar{\xi}_3[(D_H\bar{\xi}_1)(H\bar{\xi}_2)] \\
&\quad \overset{(4.13)}{=} (-1)^{(j_1+j_2)(j_3+\iota)+j_3+\iota+1}\bar{\xi}_2[(D_H\bar{\xi}_3)(H\bar{\xi}_1)] - (-1)^{j_1+\iota}\bar{\xi}_3[(D_H\bar{\xi}_1)(H\bar{\xi}_2)] \\
&\quad \overset{(4.9)}{=} (-1)^{j_2+\iota+(j_1+\iota, j_2+j_3)}\bar{\xi}_1[(D_H\bar{\xi}_2)H\bar{\xi}_3] \\
&\quad = -(-1)^{(j_1+\iota)(j_2+\iota)}\bar{\xi}_3(H(D_H\bar{\xi}_2)^*\bar{\xi}_1)
\end{aligned}
\tag{4.14}
$$

where

$$
[(D_H\bar{\xi}_2)^*\bar{\xi}_1]_p = \sum_{t,q\in I}\sum_{0\leqslant l,m\in\mathbb{Z}} \frac{(-D)^m}{m!}\left(s_q\left(m+\tfrac{1}{2}\right)(a^{\iota}_{p,t,l})D^l(\xi_{2,t})\xi_{1,q}\right) \tag{4.15}
$$

and $\bar{\xi}_i = \{\xi_{\iota,q} \mid q \in I\}$. Since $\bar{\xi}_3$ is arbitary, we have

$$
\begin{aligned}
&(-1)^{(j_1+\iota+1)(j_2+\iota)}(D_H\bar{\xi}_2)(H\bar{\xi}_1) - (-1)^{(j_1+\iota)(j_2+\iota)+(j_2+\iota+1)(j_1+\iota)}(D_H\bar{\xi}_1)(H\bar{\xi}_2) \\
&\quad = -(-1)^{(j_1+\iota)(j_2+\iota)}H[(D_H\bar{\xi}_2)^*\bar{\xi}_1].
\end{aligned}
\tag{4.16}
$$

Finally, by equations (3.27), (3.28), (4.6) and (4.16)

$$
\begin{aligned}
[H\bar{\xi}_1, H\bar{\xi}_2] &= H[(-1)^{j_1+\iota}\partial_{H\bar{\xi}_1}\bar{\xi}_2 - (-1)^{j_2+\iota+(j_1+\iota)(j_2+\iota)}\partial_{H\bar{\xi}_2}\bar{\xi}_1] \\
&\quad - (-1)^{(j_1+\iota)(j_2+\iota)}H[(D_H\bar{\xi}_2)^*\bar{\xi}_1].
\end{aligned}
\tag{4.17}
$$

Thus $H(\Omega)$ is a subalgebra of $\mathcal{L}$. $\square$

Let H_1 and H_2 be matrix differential operators of the same type ι. If $aH_1 + bH_2$ is Hamiltonian for any $a, b \in \mathbb{F}$, then we call (H_1, H_2) a *Hamiltonian pair*. For any two matrix differential operators H_1 and H_2, we define the Schouten–Nijenhuis super-bracket $[H_1, H_2] : \Omega^3 \to \tilde{\mathcal{U}}$ by

$$[H_1, H_2](\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3) = (-1)^{j_1}\bar{\xi}_3((D_{H_1}\bar{\xi}_1)H_2\bar{\xi}_2) + (-1)^{j_1}\bar{\xi}_3((D_{H_2}\bar{\xi}_1)H_1\bar{\xi}_2)$$
$$+(-1)^{j_2+(j_1+\iota, j_2+j_3)}\bar{\xi}_1((D_{H_1}\bar{\xi}_2)H_2\bar{\xi}_3) + (-1)^{j_2+(j_1+\iota, j_2+j_3)}\bar{\xi}_1((D_{H_2}\bar{\xi}_2)H_1\bar{\xi}_3)$$
$$+(-1)^{j_3+(j_3+\iota, j_1+j_2)}\bar{\xi}_2((D_{H_1}\bar{\xi}_3)H_2\bar{\xi}_1) + (-1)^{j_3+(j_3+\iota, j_1+j_2)}\bar{\xi}_2((D_{H_2}\bar{\xi}_3)H_1\bar{\xi}_1) \tag{4.18}$$

for $\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3 \in \Omega$. When the characteristic of $\mathbb{F}$ is not 2, (4.9) is equivalent to $[H, H] = 0$. In general, we have the following corollary.

Corollary 4.2. Suppose that the characteristic of $\mathbb{F}$ is not 2. Matrix differential operators H_1 and H_2 of the same type forms a Hamiltonian pair if and only if they satisfy (4.3) and

$$[H_1, H_1] = 0 \qquad [H_2, H_2] = 0 \qquad [H_1, H_2] = 0. \tag{4.19}$$

5. Examples

In this section, we shall give some examples of Hamiltonian superoperators, using the notation of section 4.

Example 1. Let

$$F(D) = \sum_{i=0}^{n} a_i D^{2i} \qquad \text{where } a_i \in \mathbb{F}. \tag{5.1}$$

We define H by

$$H(\bar{\xi}) = (-1)^i F(D)(\bar{\xi}) \qquad \text{for } \bar{\xi} \in \Omega_i. \tag{5.2}$$

Then H satisfies (4.3) and (4.9). Thus H is a Hamiltonian superoperator of type 0. The corresponding Poisson structure on $\tilde{\mathcal{U}}$ is

$$\omega_H(u, v) = (-1)^i(\bar{\delta}v)[F(D)(\bar{\delta}u)] = (-1)^i \sum_{p\in I}[(F(D)(\delta_p u))(\delta_p v)]^{\sim} \tag{5.3}$$

where $u \in \tilde{\mathcal{U}}_i$, $v \in \tilde{\mathcal{U}}$. Such a Poisson structure could be useful in proving the integrability of certain nonliner 'super-systems'. Note that, in the formal variational calculus of Gel'fand and Dikii, only odd polymials of d/dx are Hamiltonian. In our theory, only even polynomials of D are Hamiltonian. This shows an essential difference between their theory and ours.

Example 2. We use the notation of (4.2). Let

$$H^1_{p,q} = H^2_{p,q} = \sum_{l\in I} a^l_{p,q} s_l(-\tfrac{1}{2}) \qquad a^l_{p,q} \in \mathbb{F}. \tag{5.4}$$

Then H is a matrix operator of type 1. The super skew-symmetry (4.3) is equivalent to

$$a^l_{p,q} = -a^l_{q,p} \qquad \text{for } p, q, l \in I. \tag{5.5}$$

Moreover, for $\bar{\xi}_i \in \Omega_{j_i}$, $i = 1, 2, 3$,

$$\bar{\xi}_3((D_H\bar{\xi}_1)H\bar{\xi}_2) = \sum_{p,q,t,l,m\in I} [a^l_{p,q}a^m_{l,t}\xi_{1,q}s_m(-\tfrac{1}{2})\xi_{2,t}\xi_{3,p}]^{\sim}$$

$$= (-1)^{j_1+1} \sum_{p,q,t,l,m\in I} [a^l_{p,q}a^m_{l,t}s_m(-\tfrac{1}{2})\xi_{1,q}\xi_{2,t}\xi_{3,p}]^{\sim} \tag{5.6}$$

$$\bar{\xi}_1((D_H\bar{\xi}_2)H\bar{\xi}_3) = (-1)^{j_2+1} \sum_{p,q,t,l,m\in I} [a^l_{p,q}a^m_{l,t}s_m(-\tfrac{1}{2})\xi_{2,q}\xi_{3,t}\xi_{1,p}]^{\sim}$$

$$= (-1)^{j_2+(j_2+j_3)(j_1+1)+1} \sum_{p,q,t,l,m\in I} [a^l_{q,t}a^m_{l,p}s_m(-\tfrac{1}{2})\xi_{1,q}\xi_{2,t}\xi_{3,p}]^{\sim} \tag{5.7}$$

$$\bar{\xi}_2((D_H\bar{\xi}_3)H\bar{\xi}_1) = (-1)^{j_3+1} \sum_{p,q,t,l,m\in I} [a^l_{p,q}a^m_{l,t}s_m(-\tfrac{1}{2})\xi_{3,q}\xi_{1,t}\xi_{2,p}]^{\sim}$$

$$= (-1)^{j_3+(j_1+j_2)(j_3+1)+1} \sum_{p,q,t,l,m\in I} [a^l_{t,p}a^m_{l,q}s_m(-\tfrac{1}{2})\xi_{1,q}\xi_{2,t}\xi_{3,p}]^{\sim}. \tag{5.8}$$

Thus equation (4.9) is equivalent to

$$\sum_{l\in I}(a^l_{p,q}a^m_{l,t} + a^l_{q,t}a^m_{l,p} + a^l_{t,p}a^m_{l,q}) = 0 \qquad \text{for any } m, p, q, t \in I. \tag{5.9}$$

We define an operation $[\cdot,\cdot]$ on $\mathcal{S}$ by

$$[s_p, s_q] = \sum_{l\in I} a^l_{p,q}s_l \qquad p, q \in I. \tag{5.10}$$

Then equations (5.5) and (5.9) show that an operator H of the form (5.4) is Hamiltonian if and only if $(\mathcal{S}, [\cdot,\cdot])$ is a Lie algebra.

Remark 5.1. The operator (5.4) is an analogue of the one in equation (6.1) of [GDo]. In fact, we have Hamiltonian superoperators analogous to the other examples in section 6 of [GDo]. Suppose that a matrix differential operator H in the formal variational calculus [GDo] is given by

$$H_{p,q} = \sum_{i=0}^{n(p,q,i)} a_{p,q,i}(\{u^{(l)}_t\})\left(\frac{d}{dx}\right)^i \qquad p, q \in I \tag{5.11}$$

where $a_{p,q,i}(\{u^{(l)}_t\})$ are linear functions in $\{u^{(l)}_t\}$. We define an operator H' in our super formal variational calculus by

$$H'_{p,q} = \sum_{i=0}^{n(p,q,i)} a_{p,q,i}(\{l!s_t(-l-\tfrac{1}{2})\})D^i \qquad p, q \in I. \tag{5.12}$$

Then H' is a matrix differential operator of type 1. By an argument analogous to the above example, we can prove that H' is Hamiltonian if and only if H is Hamiltonian.

References

[BN] Balinskii A A and Novikov S P 1985 Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras *Soviet Math. Dokl.* **32** 228–31

[DGM] Dolan L, Goddard P and Montague P 1990 Conformal field theory of twisted vertex operators *Nucl. Phys.* B **338** 529–601

[D] DeWitt B 1992 *Supermanifolds* (Cambridge: Cambridge University Press) 2nd edn

[FFR] Feingold A J, Frenkel I B and Ries J F 1991 Spinor construction of vertex operator algebras, triality and $E_8^{(1)}$ *Contemp. Math.* **121**

[FLM] Frenkel I B, Lepowsky J and Meurman A 1988 *Vertex Operator Algebras and the Monster (Pure and Applied Mathematics Series)* (New York: Academic)

[GDi1] Gel'fand I M and Dikii L A 1975 Asymptotic behaviour of the resolvent of Sturm–Liouville equations and the algebra of the Korteweg–de Vries equations *Russian Math. Surveys* **30** (5) 77–113

[GDi2] Gel'fand I M and Dikii L A 1976 A Lie algebra structure in a formal variational calculation *Funct. Anal. Appl.* **10** 16–22

[NO] Negele J W and Orland H 1988 *Quantum Many-particle Systems* (Reading, MA: Addison-Wesley)

[GDo] Gel'fand I M and Dorfman I Ya 1979 Hamiltonian operators and algebraic structures related to them *Funct. Anal. Appl.* **13** 248–62

[O1] Marshall Osborn J Novikov algebras 1992 *Nova J. Alg. Geom.* **1** 1–14

[O2] Marshall Osborn J 1992 Simple Novikov algebras with an idempotent *Commun. Alg.* **20** 2729–53

[O3] Marshall Osborn J 1994 Infinite dimensional Novikov algebras of characteristic 0 *J. Algebra* **167** 146–67

[T] Tsukada H 1990 Vertex operator superalgebras *Commun. Math. Phys.* **18** 2249–74

[X1] Xu X 1992 On spinor vertex operator algebras and their modules *Preprint* (*Technical Report* HKUST)

[X2] Xu X 1995 Hamiltonian operators and associative algebras with a derivation *Lett. Math. Phys.* (accepted)

[Z] Zel'manov E I 1987 On a class of local translation invariant Lie algebras *Soviet Math. Dokl.* **35** 216–8